Problems :

1) Find the divergence of $\vec{F} = (x+2y-z)\hat{i} + (x-y+z)\hat{j} + (-x+y+2z)\hat{k}$.

Solution: $\vec{F} = \underbrace{(x+2y-z)}_{F_1}\hat{i} + \underbrace{(x-y+z)}_{F_2}\hat{j} + \underbrace{(-x+y+2z)}_{F_3}\hat{k}$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \cancel{1\hat{i} + (-1)\hat{j} + 2\hat{k}}$$

$$= \cancel{\hat{i} - \hat{j} + 2\hat{k}}$$

$$= 1 + (-1) + 2$$

$$= 1 - 1 + 2 = 2$$

2) Find the divergence of the following vectors

i) $\vec{F} = x^2\cos z\,\hat{i} + y\log x\,\hat{j} - yz\,\hat{k}$

ii) $\vec{F} = xy\sin z\,\hat{i} + y^2\sin x\,\hat{j} + z^2\sin xy\,\hat{k}$ at $(0, \pi/2, \pi/2)$

iii) $\vec{F} = x^3 z\,\hat{i} + y^3 x\,\hat{j} + z^3 y\,\hat{k}$

Solution: i) $\vec{F} = x^2\cos z\,\hat{i} + y\log x\,\hat{j} - yz\,\hat{k}$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2\cos z) + \frac{\partial}{\partial y}(y\log x) + \frac{\partial}{\partial z}(-yz)$$

$$= 2x\cos z + \log x - y$$

ii) $\vec{F} = xy\sin z\,\hat{i} + y^2\sin x\,\hat{j} + z^2\sin xy\,\hat{k}$ at $(0, \pi/2, \pi/2)$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(xy\sin z) + \frac{\partial}{\partial y}(y^2\sin x) + \frac{\partial}{\partial z}(z^2\sin xy)$$

$$\operatorname{div}\vec{F} = y\sin z + 2y\sin x + 2z\sin xy$$

$$\operatorname{div}\vec{F}_{(0,\pi/2,\pi/2)} = \pi/2\sin\pi/2 + 2\pi/2\sin(0) + 2\pi/2\sin(0)\pi/2$$

$$= \pi/2(1) + 0 + 0$$

$$= \pi/2$$

iii) $\vec{F} = x^3z\,\hat{i} + y^3x\,\hat{j} + z^3y\,\hat{k}$ H.W

3) Find $\nabla(\operatorname{div}\vec{F})$ if $\vec{F} = x^2\hat{i} + 3y\hat{j} + x^3\hat{k}$

Solution: $\vec{F} = x^2\hat{i} + 3y\hat{j} + x^3\hat{k}$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(x^3)$$

$$div\vec{F} = 2x + 3 + 0$$

$$= 2x+3$$

$$\nabla(div\vec{F}) = \nabla(2x+3)$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(2x+3)$$

$$= \frac{\partial}{\partial x}(2x+3)\hat{i} + \frac{\partial}{\partial y}(2x+3)\hat{j} + \frac{\partial}{\partial z}(2x+3)$$

$$= 2\hat{i}$$

4) Show that the vector field $\vec{F} = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$ is solenoidal.

Solution: Solenoidal vector, $div\vec{F} = 0$.

$$\vec{F} = \overset{F_1}{(x+3y)}\hat{i} + \overset{F_2}{(y-3z)}\hat{j} + \overset{F_3}{(x-2z)}\hat{k}$$

$$div\vec{F} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1 + 1 - 2$$

$$= 2 - 2$$

$$= 0$$

$\therefore$ Then given vector field is solenoidal.

5) Find the constant a, so that $\vec{F} = y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$ is solenoidal.

Solution: By given $\vec{F}$ is Solenoidal

$$\Rightarrow \quad \operatorname{div}\vec{F} = 0.$$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

$$\cancel{\frac{\partial}{\partial x}[x(y^2-z^2}\quad \frac{\partial}{\partial x}[y(ax^2+z)] + \frac{\partial}{\partial y}[x(y^2-z^2)] + \frac{\partial}{\partial z}[2xy(z-xy)] = 0$$

$$\Rightarrow \quad 2axy + 2xy + 2xy = 0$$

$$\Rightarrow \quad 2xy[a+1+1] = 0$$

$$\Rightarrow \quad 2xy[a+2] = 0$$

$$\Rightarrow \quad a+2 = 0$$

$$\Rightarrow \quad a = -2$$

6) If $\phi = xyz$ and $\vec{F} = x^2yz\,\hat{i} + y^2zx\,\hat{j} + z^2xy\,\hat{k}$ find $\operatorname{div}(\phi\vec{F})$

H.W

7) Show that $\nabla \cdot \left\{ \frac{f(r)}{r} \vec{r} \right\} = \frac{1}{r^2} \frac{d}{dr} \left\{ r^2 f(r) \right\}$
where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Proof: Consider, $\nabla \cdot \left\{ \frac{f(r)}{r} (x\hat{i} + y\hat{j} + z\hat{k}) \right\} \Rightarrow$

$$\nabla \cdot \left\{ \frac{f(r)}{r} x\hat{i} + \frac{f(r)}{r} y\hat{j} + \frac{f(r)}{r} z\hat{k} \right\}$$

$$\Rightarrow = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left\{ \frac{f(r)}{r} x\hat{i} + \frac{f(r)}{r} y\hat{j} + \frac{f(r)}{r} z\hat{k} \right\}$$

$$= \frac{\partial}{\partial x}\left[\frac{f(r)}{r} x \right] + \frac{\partial}{\partial y}\left[\frac{f(r)}{r} y \right] + \frac{\partial}{\partial z}\left[\frac{f(r)}{r} z \right] \rightarrow (1)$$

Consider $\frac{\partial}{\partial x}\left[\frac{f(r)}{r} x \right] = \left[\frac{f(r)}{r}(1) + f(r)\, x \left(\frac{-1}{r^2} \right) \frac{\partial r}{\partial x} + f'(r) \frac{x}{r} \frac{\partial r}{\partial x} \right]$

$$= \left[\frac{f(r)}{r} - \frac{f(r)\, x}{r^2} \frac{x}{r} + f'(r) \frac{x}{r} \frac{x}{r} \right]$$

$$= \left[\frac{f(r)}{r} - \frac{f(r)\, x^2}{r^3} + \frac{f'(r)\, x^2}{r^2} \right]$$

Illy

$$\frac{\partial}{\partial y}\left[\frac{f(r)}{r} y \right] = \left[\frac{f(r)}{r} - \frac{f(r)}{r^3} y^2 + f'(r)\, y^2/r^2 \right] \rightarrow (2)$$

$$\frac{\partial}{\partial z}\left[\frac{f(r)}{r} z \right] = \left[\frac{f(r)}{r} - \frac{f(r)}{r^3} z^2 + f'(r) \frac{z^2}{r^2} \right]$$

Sub 2 in 1.

$$\Rightarrow \frac{f(r)}{r} - \frac{1}{r^2} f(r) \frac{x^2}{r} + f'(r)\frac{x^2}{r^2} + \frac{f(r)}{r} - \frac{1}{r^2} f(r)\frac{y^2}{r} + f'(r)\frac{y^2}{r^2} + \frac{f(r)}{r} - \frac{1}{r^2} f(r)\frac{z^2}{r} + f'(r)\frac{z^2}{r^2}$$

$$= \frac{3f(r)}{r} - \frac{1}{r^2} f(r)\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + f'(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right]$$

$$= \frac{3f(r)}{r} - \frac{1}{r^2} f(r)\left[\frac{x^2+y^2+z^2}{r}\right] + f'(r)\left[\frac{x^2+y^2+z^2}{r^2}\right]$$

$$= \frac{3f(r)}{r} - \frac{f(r)}{\cancel{r^2}}\left[\frac{\cancel{r^2}}{r}\right] + f'(r)\left[\frac{\cancel{r^2}}{\cancel{r^2}}\right]$$

$$= \frac{3f(r)}{r} - \frac{f(r)}{r} + f'(r)$$

$$= \frac{2f(r)}{r} + f'(r)$$

$\times^{ly}$ and $\div$ by r^2.

$$= \frac{1}{r^2}\left[\frac{2f(r)\,r^2}{r} + f'(r)\,r^2\right]$$

$$= \frac{1}{r^2}\left[2r\,f(r) + f'(r)\,r^2\right]$$

$$= \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right]$$

$$\frac{d}{dr}\left[r^2 f(r)\right]$$

$$= 2r f(r) + f'(r)\,r^2.$$

8) If $\phi = \phi(r)$, Show that $div\{\phi(r)\vec{r}\} = 3\phi(r) + r\phi'(r)$

deduce the following (for $r \neq 0$).

i) $div\{\phi(r)\hat{r}\} = \frac{1}{r^2}\frac{d}{dr}\{r^2\phi(r)\}$ → H.W $\hat{r} = \frac{\vec{r}}{r}$

ii) $div\,\hat{r} = \frac{2}{r}$ iii) $div\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}$ → H.W

iv) $div\left\{r\,\nabla\left(\frac{1}{r^3}\right)\right\} = \frac{3}{r^4}$

Solution: Given $\phi = \phi(r)$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$div\{\phi(r)\vec{r}\} = \phi(r)\,div\,\vec{r} + \nabla\phi(r)\cdot\vec{r} \rightarrow (1)$$

$$[div(\phi\vec{F}) = \phi\,div\,\vec{F} + \nabla\phi\cdot\vec{F}]$$

$\vec{r} = \underbrace{x}_{F_1}\hat{i} + \underbrace{y}_{F_2}\hat{j} + \underbrace{z}_{F_3}\hat{k}$

$$div\,\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1 = 3 \rightarrow (2)$$

$$\nabla\phi(r) = \frac{d\phi}{dr}\frac{\vec{r}}{r} = \phi'(r)\frac{\vec{r}}{r} \rightarrow (3)$$

Sub (2) & (3) in (1)

$$div\{\phi(r)\vec{r}\} = \phi(r)\,3 + \frac{\phi'(r)}{r}(\vec{r}\cdot\vec{r})$$

$$= 3\phi(r) + \frac{\phi'(r)}{\cancel{r}}r^{\cancel{2}}$$

$$= 3\phi(r) + \phi'(r)\,r \,//$$

$\vec{r}\cdot\vec{r}$
$= (x\hat{i}+y\hat{j}+z\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})$
$= x^2+y^2+z^2$
$= r^2$

ii) $\text{div}\,\hat{r} = \dfrac{2}{r}$ $\qquad \hat{r} = \dfrac{\vec{r}}{r}$

$$\text{div}\left\{\frac{1}{r}\vec{r}\right\} = \frac{1}{r}\,\text{div}\,\vec{r} + \nabla\left(\frac{1}{r}\right)\cdot\vec{r} \rightarrow (1)$$

$$\text{div}\,\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 3 \rightarrow (2)$$

$$\nabla\left(\frac{1}{r}\right) = \frac{d}{dr}\left(\frac{1}{r}\right)\cdot\frac{\vec{r}}{r} = -\frac{1}{r^2}\cdot\frac{\vec{r}}{r} = -\frac{\vec{r}}{r^3} \rightarrow (3)$$

Sub (2) & (3) in (1)

$$\text{div}\left\{\frac{1}{r}\vec{r}\right\} = \frac{1}{r}\,3 + \left(-\frac{\vec{r}}{r^3}\right)\cdot\vec{r}$$

$$= \frac{3}{r} - \frac{(\vec{r}\cdot\vec{r})}{r^3}$$

$$= \frac{3}{r} - \frac{\cancel{r^2}}{r^{\cancel{3}}}$$

$$= \frac{3}{r} - \frac{1}{r}$$

$$= \frac{2}{r}$$

iv) $\operatorname{div}\left\{r\,\nabla\left(\frac{1}{r^3}\right)\right\} = \frac{3}{r^4}$

<u>solution:</u> Consider $\operatorname{div}\left\{r\,\nabla\left(\frac{1}{r^3}\right)\right\}$

To find $\nabla\left(\frac{1}{r^3}\right)$:

$$\nabla\left(\frac{1}{r^3}\right) = \frac{d}{dr}\left(\frac{1}{r^3}\right)\frac{\vec{r}}{r}$$

$$= -\frac{3}{r^4}\,\frac{\vec{r}}{r} = \frac{-3\vec{r}}{r^5}$$

$$\Rightarrow -3\operatorname{div}\left\{\cancel{r}\cdot\frac{\vec{r}}{r^{\cancel{5}4}}\right\} = -3\operatorname{div}\left\{\frac{\vec{r}}{r^4}\right\}$$

$$= -3\left[\frac{1}{r^4}\operatorname{div}\vec{r} + \nabla\left(\frac{1}{r^4}\right)\cdot\vec{r}\right\}$$

$$= -3\left[\frac{1}{r^4}\,3 + \frac{d}{dr}\left(\frac{1}{r^4}\right)\left(\frac{\vec{r}}{r}\cdot\vec{r}\right)\right]$$

$$= -3\left[\frac{3}{r^4} - \frac{4}{r^5}\cdot\frac{r^2}{r}\right]$$

$$= -3\left[\frac{3}{r^4} - \frac{4}{r^4}\right]$$

$$= -3\left[-\frac{1}{r^4}\right]$$

$$= 3/r^4$$